

Stochastic perturbations and fisheries management

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Abstract

As most natural resources, fisheries are affected by random disturbances. The evolution of such resources may be modelled by a succession of deterministic process and random perturbations on biomass and/or growth rate at random times. We analyze the impact of the characteristics of the perturbations on the management of natural resources. We highlight the importance of using a dynamic programming approach in order to completely characterize the optimal solution, we also present the properties of the controlled model and give the behavior of the optimal harvest for specific jump kernels.

Keywords: Piecewise Deterministic Markov Process (PDMP), optimal control, value function.

Recommendations for Resource Managers:

- In the context of updated biomass, for a centrally disturbed biomass and sufficiently high effort the optimal harvest increases with biomass jump rate
- In the context of jointly updated biomass and growth rate, for a centrally disturbed biomass and sufficiently high effort the optimal harvest increases with the biomass jump rate
- In the context of jointly updated biomass and growth rate, for sufficiently high effort the optimal harvest decreases with the growth jump rate

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1 Introduction

The evolution of natural resources is most often disturbed by random events. These disturbances occur at times that are not necessarily at regular intervals. Hence, the management of natural resources must take into account the characteristics of these disturbances.

The inclusion of stochastic perturbations in resource management has been the subject of numerous articles in the literature, [1], [6], [9], [10]. Most of these works concerns growth processes subject to perturbation continuously or at regular intervals, whereas, as mentioned above, perturbations occur most often at discrete random times. Our study tries to take into account the random nature of not only the perturbations magnitude of the resource but also the occurrence times of these perturbations.

For systems with random perturbations the state variables are updated at random times and between these random times, the state variables are governed by deterministic processes. The most appropriate framework for the study of such systems seems to be that of the Piecewise Deterministic Markov Process (PDMP) [2]. Using this framework, Hanson and Tuckwell [5] study the time to extinction of a population with some specific growth function and random perturbation structure. Applications in dynamic population are studied in [7], [8]. Hanson [4] gives a panorama of the models developed in various fields with this framework. We are interested by the optimal management of fisheries in this framework. The goal of this paper is to study the behavior of the control variable with respect to the jump rate of the perturbation process.

In Section 2 we first present the resource growth model with its deterministic and stochastic components and the corresponding PDMP framework. Secondly, we express the controlled problem with updated biomass, we highlight the importance of using a dynamic programming approach in order to completely characterize the optimal solution, we also present the properties of the controlled model and give the application for a specific jump kernel. Finally in Section 4, we consider the case of updated biomass and growth rate.

2 The model with updated biomass

We assume that the evolution of the biomass in a fishery is mainly governed by a determinist continuous process while it is observed or perturbed only at discrete random times. In absence of update, the evolution of the resource biomass $x(t)$ at time t is governed by a deterministic growth model:

$$\frac{dx(t)}{dt} = G(x(t)) - h(x(t), e(t)),$$

with initial condition : $x(0) = x_0$,

with $0 < x_0 < K$. The parameter K is the carrying capacity of the studied system and h is the harvest and $e(t)$ is the harvest rate.

We assume that G is a differentiable concave growth function such that $G(0) = 0$, $G'(x) > 0$ for $x < K$, $G'(x) < 0$ for $x > K$. The most common example is the logistic growth function $G(x) = rx(1 - x/K)$ with the growth rate r .

We consider a resource submitted to random updates of the biomass $\mathcal{Y}_1, \mathcal{Y}_2, ..$ at random times $\tau_1, \tau_2, ..$. We assume that updates occur in a Poisson process i.e. that updates occur independently of one another and randomly in time. The distribution of times between successive updates is an exponential distribution with mean $1/\lambda$:

$$F(x) = 1 - e^{-\lambda x},$$

with the constant jump rate λ . For each random time τ_i , the biomass is updated:

$$x(\tau_i^+) = \mathcal{Y}_i \stackrel{d}{\sim} \mathcal{L}(\cdot | x(\tau_i)), \text{ at time } \tau_i, \text{ for } i \geq 1.$$

where \mathcal{L} is a conditional distribution.

Hence the dynamics of the biomass can be described by a Piecewise Deterministic Markov Process (PDMP) ([2]). The random jump process is described by the jump kernel operator Q . To each function θ the

operator Q associates the function $Q[\theta]$ which is defined by: $Q[\theta](x) = \int_{\mathcal{Y}} \theta(\mathcal{Y}) d\mathcal{L}(\mathcal{Y}|x)$, $Q[\theta]$ is assumed continuous and Lipschitz for all $\theta \in \mathcal{C}(\Omega)$.

For example: $\mathcal{Y}_i = \mathcal{Z}_i x(\tau_i)$ with $\mathcal{Z}_i \stackrel{d}{\sim} \mathcal{U}(\underline{z}, \bar{z})$, hence the associated jump kernel Q is:

$$Q[\theta](x) = \frac{1}{\bar{z} - \underline{z}} \int_{\underline{z}}^{\bar{z}} \theta(zx) dz = \frac{1}{(\bar{z} - \underline{z})x} \int_{\underline{z}x}^{\bar{z}x} \theta(y) dy.$$

2.1 The biomass growth process

In order to describe the biomass growth process, we define the function $X(t; x, \tau)$ at time t . If the biomass was x at time τ , the evolution of $X(t; x, \tau)$ is given by $(\mathcal{S}_{x, \tau})$:

$$\frac{dX(t; x, \tau)}{dt} = g(X(t; x, \tau), e(t)) \equiv G(X(t; x, \tau)) - h(X(t; x, \tau), e(t)),$$

with initial condition: $X(\tau; x, \tau) = x$.

Knowing these characteristics, denoting τ_0 at $x_0^+ = x_0$, the biomass growth process $\{X_t(x_0) : t \geq 0\}$ starting with biomass x_0 at initial time, may be expressed, for $i \geq 1$:

$$X_t = X(t; x_{i-1}^+, \tau_{i-1}), \quad \tau_{i-1} < t \leq \tau_i,$$

$$\text{while at time } \tau_i, x_i^+ \stackrel{d}{\sim} \mathcal{L}(\cdot | X_{\tau_i}).$$

The process $\{X_t(x_0) : t \geq 0\}$ starting at x_0 is composed of the successive curves $X(t; x_0, 0)$, $X(t; x_{\tau_1}^+, \tau_1)$, ..., $X(t; x_{\tau_i}^+, \tau_i)$, This process depends on the successive random updated time τ_i and random jump at corresponding τ_i .

Remark: by composite construction, for $\tau_{i-1} < s < t < \tau_i$, we have: $X(t; X(s; x, \tau_{i-1}), s) = X(t; x, \tau_{i-1})$ and $X(t; x, \tau) = X(t - \tau; x, 0)$.

2.2 The control problem

Given a biomass x and an effort e , the instantaneous gain of consumption $l(x, e)$ is determined. Therefore, we assume a regulator maximizing expected discounted gain on an infinite horizon:

$$J(x_0, e(\cdot)) = E \left[\int_0^{+\infty} l(X_t(x_0), e(t)) e^{-\delta t} dt \right], \quad (1)$$

with the $X_t(x_0)$ solution obtained with successive systems $(\mathcal{S}_{x_0,0}), (\mathcal{S}_{x_{\tau_1}^+, \tau_1}), \dots$. The expectation in Equation (1) is related to the successive random updated time τ_i and random jump at corresponding τ_i . The effort e is subject to the constraints: $0 \leq e(t) \leq \bar{e}$ for all $t > 0$. The instantaneous gain is assumed proportional to the effort: $l(x, e) = l_0(x)e$. Thus we consider the function value V defined by: $V(x) = \max_{e(\cdot) \in [0, \bar{e}]} J(x, e(\cdot))$. Assuming $V \in \mathcal{C}^1([0, K])$, we can formally deduce (see Appendix A, with restrictive conditions [3], [2] gives mathematical justification) the Bellman Hamilton Jacobi (BHJ) equation:

$$\max_{e \in [0, \bar{e}]} [V'(x)g(x, e) - (\delta + \lambda)V(x) + l(x, e) + \lambda Q[V](x)] = 0. \quad (2)$$

The harvest is assumed proportional to the effort e i.e. $h(x, e) = h_0(x)e$, the BHJ equation becomes:

$$\max_{e \in [0, \bar{e}]} [l_0(x) - h_0(x)V'(x)]e + V'(x)G(x) - (\delta + \lambda)V(x) + \lambda Q[V](x) = 0. \quad (3)$$

In Equation (3), the effort e depends on the biomass X . This highlights the existence of a critical value x^* solution of:

$$l_0(x) - h_0(x)V'(x) = 0. \quad (4)$$

As $0 \leq e \leq \bar{e}$, the optimal effort is a feedback control $e^*(t) = \mathcal{E}(X_t)$ where the function \mathcal{E} is defined by:

$$\mathcal{E}(x) = \begin{cases} 0, & \text{if } l_0(x) - h_0(x)V'(x) < 0, \\ \frac{G(x)}{h_0(x)}, & \text{if } l_0(x) - h_0(x)V'(x) = 0, \\ \bar{e}, & \text{if } l_0(x) - h_0(x)V'(x) > 0, \end{cases}$$

where the function value V is defined by:

$$[l_0(x) - h_0(x)V'(x)]_+ \bar{e} + V'(x)G(x) - (\delta + \lambda)V(x) + \lambda Q[V](x) = 0. \quad (5)$$

But Equation (4) is not sufficient to characterize the critical values. By using a dynamic programming equation, we obtain a complementary condition based on Euler-Lagrange condition.

2.3 Euler-Lagrange condition

The value function $V(x_0)$ is the solution to the optimization problem:

$$V(x_0) = J(x_0, e^*(\cdot)) = \max_{e(\cdot) \in [0, \bar{e}]} E \left[\int_0^{+\infty} l(X_t(x_0), e(t)) e^{-\delta t} dt \right],$$

with $X_t(x_0)$ solution obtained with successive systems $(\mathcal{S}_{x_0, 0}), (\mathcal{S}_{x_{\tau_1}^+, \tau_1}), \dots$

Using the strong Markov property, we may express [2]:

$$V(x_0) = J(x_0, e^*(\cdot)) = E_\tau \left[\int_0^\tau l(X(t; x_0, 0), e^*(t)) e^{-\delta t} dt + Q[V](X(\tau; x_0, 0)) e^{-\delta \tau} \right].$$

We consider the first term in the right hand:

$$\begin{aligned} E_\tau \left[\int_0^\tau l(X(t; x_0, 0), e^*(t)) e^{-\delta t} dt \right] &= \lambda \int_0^{+\infty} \int_0^\tau l(X(t; x_0, 0), e^*(t)) e^{-\delta t} dt e^{-\lambda \tau} d\tau \\ &= \int_0^{+\infty} l(X(t; x_0, 0), e^*(t)) e^{-(\delta + \lambda)t} dt, \end{aligned}$$

when by inverting integration with respect to t and τ Finally we obtain the dynamic programming equation:

$$V(x_0) = \max_{e(\cdot) \in [0, \bar{e}]} \int_0^{+\infty} [l(X(t; x_0, 0), e(t)) + \lambda Q[V](X(t; x_0, 0))] e^{-(\delta + \lambda)t} dt.$$

To simplify the expressions, we denote $X(t, x_0) \equiv X(t; x_0, 0)$. We have the opportunity, as in the deter-

ministic control case, to deduce the expression of the effort $e(t)$ in terms of the biomass:

$$l(X(t, x_0), e(t)) = l_0(X(t, x_0))e(t) = \frac{l_0}{h_0}(X(t, x_0)(G(X(t, x_0)) - \dot{X}(t, x_0)))$$

and then we obtain the new form of the objective. Thus the optimization problem becomes:

$$V(x_0) = \max_{X(\cdot) \in \mathcal{C}_{x_0}} \int_0^{+\infty} \left[\frac{l_0}{h_0}(X(t, x_0))(G(X(t, x_0)) - \dot{X}(t, x_0)) + \lambda Q[V](X(t, x_0)) \right] e^{-(\delta+\lambda)t} dt.$$

\mathcal{C}_{x_0} being the set of admissible curves:

$$\mathcal{C}_{x_0} = \{X(\cdot) \in BC^1([0, K]), X(0) = x_0, G(X(t)) - h_0(X(t))\bar{e} \leq \dot{X}(t) \leq G(X(t))\},$$

with BC^1 stands for the bounded with bounded derivative function defined on the interval $[0, K]$. We deduce:

Proposition 2.1 *Assuming that V and $Q[V] \in BC^1([0, K])$, a critical value x^* is solution of the system of equations:*

$$\begin{aligned} l_0(x) - h_0(x)V'(x) &= 0, \\ (\delta + \lambda - G'(x)) \left[\frac{l_0}{h_0} \right](x) &= \left[\frac{l_0}{h_0} \right]'(x)G(x) + \lambda[Q[V]]'(x), \end{aligned} \quad (6)$$

where the value fonction V is solution of Equation (5).

Proof: We have an implicit problem of Calculus of Variations. $X(\cdot)$ stands for an interior solution, let $\mathcal{L}(\cdot, \cdot)$ the non actualized integrand: $\mathcal{I}(X, \dot{X}) = \frac{l_0}{h_0}(X)(G(X) - \dot{X}) + \lambda Q[V](X)$, then $X(\cdot)$ has to satisfy the Euler Lagrange condition:

$$\mathcal{I}_X(X(t), \dot{X}(t)) = \frac{d}{dt} \mathcal{I}_{\dot{X}}(X(t), \dot{X}(t)) - (\delta + \lambda) \mathcal{I}_{\dot{X}}(X(t), \dot{X}(t)).$$

The Euler Lagrange condition enhances:

$$\left[\frac{l_0}{h_0}\right]'(X(t))G(X(t)) + \frac{l_0}{h_0}(X(t))G'(X(t)) + \lambda[Q[V]]'(X(t)) = (\delta + \lambda)\left[\frac{l_0}{h_0}\right](X(t)).$$

The differential equation is reduced to an algebraic Equation (6). □

Let $x^*(\lambda)$ the lower critical value, in order to avoid scaling of function V (V is defined by Equation (5) up to a multiplicative constant for $x < x^*(\lambda)$), by using Equations (4) and (6) becomes:

$$\left(\delta - G'(x) + \lambda\left(1 - \frac{[Q[V]]'(x)}{V'(x)}\right)\right)\left[\frac{l_0}{h_0}\right](x) = \left[\frac{l_0}{h_0}\right]'(x)G(x). \quad (7)$$

Similarly to the standard optimal control problem (without update) the optimal effort e is given by a function \mathcal{E} of the biomass X . But this function \mathcal{E} depends on the jump rate λ by the intermediate of $x^*(\lambda)$.

2.4 The value function and the optimal control

We now analyze the behavior of the value function V at critical value x^* . Let $A(x) = l_0/h_0(x) - V'(x)$. Then, for x such that $A(x) \neq 0$ we may define $A'(x) = \left[\frac{l_0}{h_0}\right]'(x) - V''(x)$ and:

$$V''(x) = \left[\frac{l_0}{h_0}\right]'(x) - A'(x), \quad (8)$$

so we deduce the regularity of the function value with respect to biomass x :

Proposition 2.2 *The function value V is continuously twice differentiable and:*

$$V''(x^*) = \left[\frac{l_0}{h_0}\right]'(x^*), \quad (9)$$

and with respect to jump rate λ :

Proposition 2.3 *For a sufficiently small value of jump rate λ , assuming $l_0(x) = pqx - c$ and $h_0(x) = qx$ with price p , catchability q and cost c , the critical value x^* is an increasing (respectively decreasing)*

function with respect to λ if $[Q[V]]'(x^*) - V'(x^*) > 0$ (respectively < 0). Moreover the function value V and V'_x are continuously differentiable with respect to jump rate λ .

Proofs of Propositions are given in Appendix B. In the following section, we will illustrate for a concrete case, with a specific jump kernel, by a study of the sign of $[Q[V]]'(x^*) - V'(x^*)$. From V'_x continuously differentiable with respect to jump rate λ , for a sufficiently small jump rate λ , Equations (5) and (6) has a unique solution x^* so the function \mathcal{E} is given by:

$$\mathcal{E}(x) = \begin{cases} 0, & \text{if } x < x^*, \\ \frac{G(x)}{h_0(x)}, & \text{if } x = x^*, \\ \bar{e}, & \text{if } x > x^*, \end{cases}$$

and finally:

Proposition 2.4 *For a sufficiently small value of jump rate λ , assuming $l_0(x) = pqx - c$ and $h_0(x) = qx$ with price p , catchability q and cost c , the value function is not three times differentiable, more precisely at $\lambda = 0$:*

$$h_0(x^*)[x^2V''(x)]'_+(x^*) = -\frac{x^*\Sigma(x^*)}{\bar{e} - \mathcal{E}(x^*)} < 0 < h_0(x^*)[x^2V''(x)]'_-(x^*) = \frac{x^*\Sigma(x^*)}{\mathcal{E}(x^*)},$$

where $\Sigma(x) = -x^2V''(x)\left[\frac{G(x)}{x}\right]' + (\delta - G'(x))(V'(x) + V''(x)x) - G''(x)V'(x)x$ and $\Sigma(x^*) > 0$.

We now consider the growth function: $G(x) = rx(1 - x/K)$. We may deduce the behavior of the critical value with respect to growth rate r :

Proposition 2.5 *For a sufficiently small value of jump rate λ_x and λ_r , assuming $l_0(x) = pqx - c$ and $h_0(x) = qx$, the critical value $x^*(r)$ is an increasing function with respect to growth rate r .*

In Figure 1, we give an example of optimal evolution of the biomass x . The dash line represents the level of the critical value x^* .

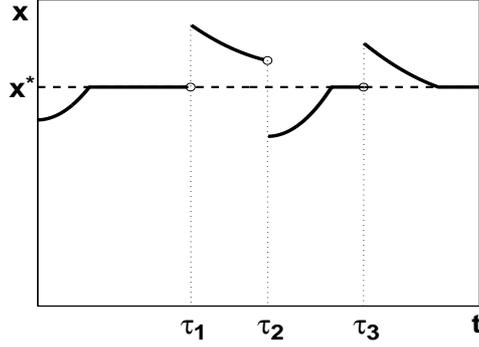


Figure 1: Optimal evolution of biomass x with biomass updated

2.5 Application to specific jump kernel

Proposition 2.6 *We assume that the updated biomass \mathcal{Y}_i is given by: $\mathcal{Y}_i = x(\tau_i)(1 + \mathcal{Z}_i\epsilon)$ where \mathcal{Z}_i follows the distribution \mathcal{H} (symmetric and centered in 0), for a sufficiently small value of jump rate λ_x and x^* not close to K .*

(i) *if $E[\mathcal{Z}] \neq 0$ then the critical value is increasing (respectively decreasing) with respect to jump rate λ if $E[\mathcal{Z}] > 0$ (respectively < 0).*

(ii) *if $E[\mathcal{Z}] = 0$ then the critical value is increasing (respectively decreasing) with respect to jump rate λ if $\mathcal{E}(x^*)$ is smaller (respectively larger) than $\frac{\bar{\epsilon}}{2}$.*

More precisely in the latter case, with the growth function $G(x) = rx(1 - x/K)$, the critical value is increasing (respectively decreasing) with respect to jump rate λ if $x^ > K(1 - \bar{\epsilon}q/(2r))$ (respectively $<$).*

Proofs are given in Appendix B. The given result for $E[\mathcal{Z}] \neq 0$ is not surprising: for instance if $E[\mathcal{Z}] > 0$, higher jump rate leads to higher biomass, hence higher possible harvest. If $E[\mathcal{Z}] = 0$, the result is more difficult to explain: for a sufficiently large value of $\bar{\epsilon}$, it is optimal to use a higher level of critical value for the biomass.

3 The model with updated biomass and growth rate

We now consider a resource submitted two types of random updates (biomass updates \mathcal{Y} , growth rate updates \mathcal{R}) at random times:

- the times between two biomass updates follows exponential distribution with mean $1/\lambda_x$
- the times between two growth rate updates follows exponential distribution with mean $1/\lambda_r$.

Hence the random update time τ between two updates follows an exponential distribution with mean $1/(\lambda_x + \lambda_r)$. For each random time τ , the biomass or the growth rate is updated:

$$x(\tau^+) = \mathcal{Y} \stackrel{d}{\sim} \mathcal{L}_x(\cdot|x(\tau)) \text{ with probability } \frac{\lambda_x}{\lambda_x + \lambda_r}$$

$$\text{and } r(\tau^+) = \mathcal{R} \stackrel{d}{\sim} \mathcal{L}_r(\cdot|r(\tau)) \text{ with probability } \frac{\lambda_r}{\lambda_x + \lambda_r},$$

where \mathcal{L}_x and \mathcal{L}_r are conditional distributions.

We assume that between two random updates the growth rate does not change, i.e. the grow is piece-wise.

The dynamics of the growth rate is:

$$\dot{r}(t) = 0.$$

As in the previous case, the dynamics of the biomass can be described by a Piecewise Deterministic Markov Process (PDMP). The random jump process are described by the jump kernels Q_x and Q_r . To each function θ of biomass x and growth rate r , the functions $Q_x[\theta]$ and $Q_r[\theta]$ are defined by: $Q_x[\theta](x, r) = \int_{\mathcal{Y}} \theta(\mathcal{Y}, r) d\mathcal{L}_x(\mathcal{Y}|x)$, $Q_r[\theta](x, r) = \int_{\mathcal{R}} \theta(x, \mathcal{R}) d\mathcal{L}_r(\mathcal{R}|r)$.

3.1 The biomass growth process

In order to describe the biomass growth process we now define the function $X(r, t; x, \tau)$ at time t . If the biomass was x at time τ , the evolution of $X(r, t; x, \tau)$ is given by $(\mathcal{S}_{x,r,\tau})$:

$$\frac{dX(r, t; x, \tau)}{dt} = G(X(r, t; x, \tau), r) - h(X(r, t; x, \tau), e(t)),$$

with initial condition: $X(r, \tau; x, \tau) = x$.

The process of the biomass $\{X_t(x_0, r_0) : t \geq 0\}$ starting at time $\tau_0 = 0$ with $x_0^+ = x_0$, may be expressed for $i \geq 1$:

$$X_t = X(r_{i-1}, t; x_{i-1}^+, \tau_{i-1}), \tau_{i-1} < t \leq \tau_i,$$

where at time $\tau_i : r_i = r_{i-1}$, $x_i^+ \stackrel{d}{\sim} \mathcal{L}_x(\cdot | X_{\tau_i})$ with probability $\frac{\lambda_x}{\lambda_x + \lambda_r}$
 and $x_i^+ = X_{\tau_i}$, $r_i \stackrel{d}{\sim} \mathcal{L}_r(\cdot | r_{i-1})$ with probability $\frac{\lambda_r}{\lambda_x + \lambda_r}$.

Given a biomass x and an effort e , we assume a regulator maximizing expected discounted gain on an infinite horizon:

$$J(x_0, r_0, e(\cdot)) = E \left[\int_0^{+\infty} l(X_t(x_0, r_0), e(t)) e^{-\delta t} dt \right],$$

with $X_t(x_0, r_0)$ solution obtained with successive systems $(\mathcal{S}_{x_0, r_0, 0}), (\mathcal{S}_{x_{\tau_1}^+, r_{\tau_1}, \tau_1}), \dots$. Thus we consider the function value V defined by: $V(x, r) = \max_{e(\cdot) \in [0, \bar{e}]} J(x, r, e(\cdot))$.

Using the same formalism than for the updated biomass model and assuming $V \in \mathcal{C}^1([0, K] \star [\underline{r}, \bar{r}])$, denoting $\lambda.Q[V] = \lambda_x Q_x[V] + \lambda_r Q_r[V]$, we can formally deduce the corresponding Bellman Hamilton Jacobi (BHJ) Equation:

$$\max_{e \in [0, \bar{e}]} [V'_x(x, r)g(x, r, e) - (\delta + \lambda_x + \lambda_r)V(x, r) + l(x, e) + \lambda.Q[V](x, r)] = 0. \quad (10)$$

The BHJ Equation becomes:

$$\max_{e \in [0, \bar{e}]} [l_0(x) - h_0(x)V'_x(x, r)]e + V'_x(x, r)G(x, r) - (\delta + \lambda_x + \lambda_r)V(x, r) + \lambda.Q[V](x, r) = 0.$$

In this Equation, the effort e depends on the biomass x and the growth rate r . As $0 \leq e \leq \bar{e}$, the optimal effort is a feedback control $e^*(t) = \mathcal{E}(X_t, R_t)$ where the function \mathcal{E} is defined by:

$$\mathcal{E}(x, r) = \begin{cases} 0, & \text{if } l_0(x) - h_0(x)V'_x(x, r) < 0, \\ \frac{G(x, r)}{h_0(x)}, & \text{if } l_0(x) - h_0(x)V'_x(x, r) = 0, \\ \bar{e}, & \text{if } l_0(x) - h_0(x)V'_x(x, r) > 0. \end{cases}$$

Hence:

$$[l_0(x) - h_0(x)V'_x(x, r)]_+ \bar{e} + V'_x(x, r)G(x, r) - (\delta + \lambda_x + \lambda_r)V(x, r) + \lambda.Q[V](x, r) = 0. \quad (11)$$

The critical value $x^*(r)$ is the solution of the equation:

$$l_0(x) - h_0(x)V'_x(x, r) = 0. \quad (12)$$

But this equation is not sufficient to characterize the critical value. By using a dynamic programming equation, we obtain a complementary condition based on Euler-Lagrange condition.

The value function $V(x_0, r_0)$ is the solution to the optimization problem:

$$V(x_0, r_0) = J(x_0, r_0, e^*(\cdot)) = \max_{e(\cdot) \in [0, \bar{e}]} E \left[\int_0^{+\infty} l(X_t(x_0, r_0), e(t)) e^{-\delta t} dt \right],$$

with $X_t(x_0, r_0)$ solution of the system $(\mathcal{S}_{x_0, r_0, 0})$.

Using the same reasoning than in the previous section, we obtain the dynamic programming equation:

$$V(x_0, r_0) = \max_{e(\cdot) \in [0, \bar{e}]} \int_0^{+\infty} [l(X(r_0, t; x_0, 0), e(t)) + \lambda.Q[V](X(r_0, t; x_0, 0), r_0)] e^{-(\delta + \lambda_x + \lambda_r)t} dt.$$

Proposition 3.1 *Assuming that $V, Q_x[V]$ and $Q_r[V] \in BC^1([0, K] \star [\underline{r}, \bar{r}])$, a critical value $x^*(r)$ is solu-*

tion of the system of equations:

$$l_0(x) - h_0(x)V'_x(x, r) = 0$$

$$\text{and } (\delta + \lambda_x + \lambda_r - G'_x(x, r)) \left[\frac{l_0}{h_0} \right] (x) = \left[\frac{l_0}{h_0} \right]' (x) G(x, r) + [\lambda \cdot Q[V]]'(x), \quad (13)$$

where the value function V is solution of the Equation (11).

For fixed r , using the same reasoning than in the previous section:

$$(\delta - G'_x(x, r) + \lambda_x(1 - \frac{[Q_x[V]]'_x(x, r)}{V'_x(x, r)}) + \lambda_r(1 - \frac{[Q_r[V]]'_x(x, r)}{V'_x(x, r)})) \left[\frac{l_0}{h_0} \right] (x) = \left[\frac{l_0}{h_0} \right]' (x) G(x, r)$$

and replacing λ by λ_x (respectively λ_r) and $\lambda Q[V]$ by $\lambda_x Q_x[V]$ (respectively $\lambda_r Q_r[V]$) we can obtain the equivalent of the Propositions 2.2 and 2.3.

Proposition 3.2 *The function value V is twice differentiable with respect to biomass x and:*

$$V''_{x^2}(x^*(r), r) = \left[\frac{l_0}{h_0} \right]' (x^*(r)). \quad (14)$$

Assuming $l_0(x) = pqx - c$ and $h_0(x) = qx$ with price p , catchability q and cost c , the critical value $x^*(r)$ is an increasing (respectively decreasing) function with respect to biomass jump rate λ_x if $[Q_x[V]]'_x(x^*(r), r) - V'_x(x^*(r), r) > 0$ (respectively < 0) and is an increasing (respectively decreasing) function with respect to growth jump rate λ_r if $[Q_r[V]]'_x(x^*(r), r) - V'_x(x^*(r), r) > 0$ (respectively < 0). Moreover the function value V and V'_x are continuously differentiable with respect to jump rate λ_x , and jump growth rate λ_r .

In the following section, we will illustrate for a concrete case, with a specific jump kernels for biomass (respectively growth rate), by a study of the sign of $[Q_x[V]]'_x(x^*, r) - V'_x(x^*, r)$ (respectively $[Q_r[V]]'_x(x^*, r) - V'_x(x^*, r)$). For each growth rate r and for sufficiently small values of jump rate λ_x, λ_r , Equations (12) and

(13) have a unique solution $x^*(r)$, the function \mathcal{E} is given by:

$$\mathcal{E}(x, r) = \begin{cases} 0, & \text{if } x < x^*(r), \\ \frac{G(x, r)}{h_0(x)}, & \text{if } x = x^*(r), \\ \bar{e}, & \text{if } x > x^*(r), \end{cases}$$

Remark 3.1 *The expression of $\mathcal{E}(x, r)$ has the same formulation than in the case without updates but due to the difference between the Propositions 2.1 and 3.1, the corresponding $x^*(r)$ differs.*

Proposition 3.3 *The value function V is twice differentiable but not third differentiable and at $\lambda = 0$:*

$$V''_{xr}(x^*(r), r) = 0, \quad (15)$$

$$V'''_{x^2r}{}^-(x^*(r), r) = -\frac{\Sigma_1(x^*(r), r)}{\mathcal{E}(x^*(r), r)} < 0 < V'''_{x^2r}{}^+(x^*(r), r) = \frac{\Sigma_1(x^*(r), r)}{\bar{e} - \mathcal{E}(x^*(r), r)} \quad (16)$$

and where $\Sigma_1(x, r) = \frac{\delta}{r} \frac{l_0}{h_0^2}(x)$. The third derivatives of the function value V with respect to biomass x and growth rate r are linked by:

$$V'''_{x^2r}{}^\pm(x^*(r), r)x^{*'}(r) + V'''_{xr^2}{}^\pm(x^*(r), r) = 0. \quad (17)$$

Proofs are given in Appendix C. In Figure 2, we give an example of optimal evolution of the biomass x with alternatively biomass and growth rate updating. The dash lines represent the successive levels of the critical value $x^*(r)$,

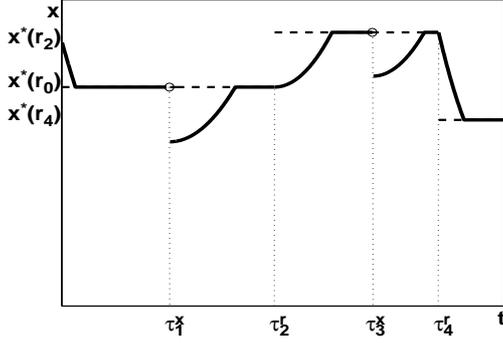


Figure 2: Optimal evolution of biomass x with biomass and growth rate updated

3.2 Application to specific jump kernels

Proposition 3.4 Assuming the updated biomass \mathcal{Y}_i (respectively, the updated growth rate \mathcal{R}_i) given by: $\mathcal{Y}_i = x(\tau_i^x)(1 + \mathcal{Z}_i\epsilon)$ (respectively, $\mathcal{R}_i = r(\tau_i^r)(1 + \mathcal{S}_i\xi)$) where \mathcal{Z}_i (respectively, \mathcal{S}_i) follows the distribution \mathcal{H}_x (respectively, \mathcal{H}_r).

For a sufficiently small value of biomass jump rate λ_x and $x^*(r)$ not close to K :

- (i) if $E[\mathcal{Z}] \neq 0$ then the critical value $x^*(r)$ is increasing (respectively, decreasing) with respect to biomass jump rate λ_x if $E[\mathcal{Z}] > 0$ (respectively, < 0).
- (ii) if $E[\mathcal{Z}] = 0$ then the critical value $x^*(r)$ is increasing (respectively, decreasing) with respect to biomass jump rate λ_x if $\mathcal{E}(x^*(r), r)$ is larger (respectively, smaller) than $\frac{\bar{e}}{2}$.

For a sufficiently small value of growth jump rate λ_r and r not close to \underline{r} and \bar{r} :

- (iii) the critical value $x^*(r)$ is increasing (respectively, decreasing) with respect to growth jump rate λ_r if $\mathcal{E}(x^*(r), r)$ is larger (respectively, smaller) than $\frac{\bar{e}}{2}$.

We deduce the following properties:

Corollary 3.1 (i) The behavior of the critical value with respect to biomass jump rate λ_x is of the same type that in the previous case with update of the biomass.

(ii) If $E[\mathcal{Z}] = 0$, the behavior of the critical value $x^*(r)$ is reversed with respect the biomass jump rate λ_x

and the growth jump rate λ_r .

(iii) The behavior of the critical value $x^*(r)$ with respect to growth jump rate λ_r is independent of the expectation $E[\xi]$.

4 Conclusion

In this article, we consider the evolution of a fishery following a continuous process and submitted to random updates at random times, we present the appropriate PDMP framework. We express the control problem with biomass updates, we highlight the importance of using a dynamic programming approach in order to completely characterize the critical value of the control. We give conditions which permit to deduce the behavior of the optimal control effort with respect to jump rate. An application to a specific jump kernel shows the possible variety of behavior of the optimal effort with respect to the random structure. For a centrally disturbed biomass and sufficiently high effort the optimal harvest increases with biomass jump rate. Finally we study the more complex case for which biomass and growth rate in the dynamics are updated. For a centrally disturbed biomass and sufficiently high effort the optimal harvest increases with the biomass jump rate and decreases with the growth jump rate.

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Appendix A

Let the value function defined by: $V(x) = \max_{e(\cdot) \in [0, \bar{e}]} J(x, e(\cdot))$. Consider a time $t > 0$, by the strong Markov property, the criteria satisfies:

$$V(x) = J(x, e^*(\cdot)) = E_{\tau} \left[\left(\int_0^{\tau} l(X(u, x), e^*(u)) e^{-\delta u} du + Q[V](X(\tau, x)) e^{-\delta \tau} \right) I_{\tau < t} + \left(\int_0^t l(X(u, x), e^*(u)) e^{-\delta u} du + V(X(t, x)) e^{-\delta t} \right) I_{t < \tau} \right],$$

$$E_{\tau} \left[\int_0^{\tau} l(X(u, x), e^*(u)) e^{-\delta u} du I_{\tau < t} \right] = \lambda \int_0^t \int_0^{\tau} l(X(u, x), e^*(u)) e^{-\delta u} du e^{-\lambda \tau} d\tau$$

then by inverting integration with respect to t and τ

$$= \int_0^t l(X(u, x), e^*(u)) (e^{-(\delta+\lambda)u} - e^{-\lambda t - \delta u}) du,$$

and hence, rearranging the terms:

$$V(x) = \max_{e(\cdot) \in [0, \bar{e}]} \left[\int_0^t [l(X(\tau, x), e(\tau)) + \lambda Q[V](X(\tau, x))] e^{-(\delta+\lambda)\tau} d\tau + e^{-(\delta+\lambda)t} V(X(t, x)) \right],$$

where $V(x)$ is independent of t , hence, formally differentiating with respect to t :

$$\max_{e \in [0, \bar{e}]} [V'(x)g(x, e) - (\delta + \lambda)V(x) + l(x, e) + \lambda Q[V](x)] = 0.$$

Appendix B

Proof of Proposition 2.2. In $\Omega - \{x | A(x) = 0\}$, the value function is smooth and is solution of:

$$\eta A(x)h_0(x)\bar{e} + V'(x)G(x) - (\delta + \lambda)V(x) + \lambda Q[V](x) = 0,$$

with $\eta = 0$ if $A(x) < 0$ and $\eta = 1$ if $A(x) > 0$.

By differentiation:

$$\eta(A'(x)h_0(x) + A(x)h_0'(x))\bar{e} + V''(x)G(x) - (\delta + \lambda - G'(x))V'(x) + \lambda[Q[V]]'(x) = 0. \quad (18)$$

Let $A'_-(x^*)$ and $A'_+(x^*)$ (respectively, $V''_-(x^*)$ and $V''_+(x^*)$) the left and right derivatives of A (resp V') at the critical value x^* (if $A(x)(x - x^*) > 0$ in the vicinity of x^* and the reverse if not). From Equations (6) and (4), using $A(x^*) = 0$, at the critical value x^* :

$$V''_-(x^*)G(x^*) - \left[\frac{l_0}{h_0} \right]'(x^*)G(x^*) = 0$$

and $A'_+(x^*)h_0(x^*)\bar{e} + V''_+(x^*)G(x^*) - \left[\frac{l_0}{h_0} \right]'(x^*)G(x^*) = 0,$

hence respectively using the Equation (8):

$$\begin{aligned} -A'_-(x^*)G(x^*) &= 0 \\ \text{and } A'_+(x^*)(h_0(x^*)\bar{e} - G(x^*)) &= 0, \end{aligned}$$

then $A'_-(x^*) = A'_+(x^*) = 0$, so A is differentiable and V is twice continuously differentiable. \square

Proof of Proposition 2.3. In order to determine the behavior of the critical value x^* in the vicinity of $\lambda = 0$, we differentiate the Equation (7) with respect to jump rate λ to obtain:

$$\left[-G''(x^*)\frac{l_0}{h_0}(x^*) + (\delta - 2G'(x^*))\left[\frac{l_0}{h_0}\right]'(x^*) - \left[\frac{l_0}{h_0}\right]''(x^*)G(x^*) \right] x^{*\prime}_\lambda(0) = \frac{l_0}{h_0}(x^*)\left(\frac{[Q[V]]'_x}{V'_x}(0, x^*) - 1\right).$$

Using expression of $l_0(x)$ and $h_0(x)$ the second equation becomes:

$$\left[-G''(x^*)\frac{l_0}{h_0}(x^*) + (\delta + 2\left(\frac{G(x^*)}{x^*} - G'(x^*)\right))\left[\frac{l_0}{h_0}\right]'(x^*) \right] x^{*\prime}_\lambda(0) = \frac{l_0}{h_0}(x^*)\left(\frac{[Q[V]]'_x}{V'_x}(0, x^*) - 1\right).$$

From $G''(x) < 0$ and $G(0) = 0$, we deduce that $G(x) - xG'(x) \geq 0$ and the left term in the brackets is positive hence $x^{*\prime}_\lambda(0)$ and $[Q[V]]'(x) - V'(x)$ for $\lambda = 0$ has the same sign.

We derive Equations (5) for $x < x^*(\lambda)$ and (4) with respect to jump rate λ at $\lambda = 0$, by using Equation (18):

$$\begin{aligned} V''_{x\lambda}(0, x^*)G(x^*) + Q[V](0, x^*) - V(0, x^*) &= \delta V'_\lambda(0, x^*) \\ \text{and } \left[\frac{l_0}{h_0}\right]'(x^*)x^{*\prime}_\lambda(0) &= V''_{x\lambda}(0, x^*). \end{aligned}$$

Hence: V and V'_x are continuously differentiable with respect to jump rate λ . \square

Proof of Proposition 2.4. By differentiation of Equation (18) multiplied by x , for $\lambda = 0$:

$$\eta[x(A'(x)h_0(x) + A(x)h'_0(x))]' \bar{e} + [x^2V''(x)]' \frac{G(x)}{x} - \Sigma(x) = 0.$$

Hence due to $A(x^*) = A'(x^*) = 0$:

$$[x^2V''(x)]'_-(x^*)G(x^*) = x^*\Sigma(x^*)$$

$$\text{and } [x^2V''(x)]'_+(x^*)(G(x^*) - h_0(x^*)\bar{e}) + (x^* \left[x \left[\frac{l_0}{h_0} \right]'(x) h_0(x) \right]'(x^*) - x^{*3}V''(x^*) \left[\frac{h_0}{x} \right]'(x^*))\bar{e} = x^*\Sigma(x^*).$$

From expression of l_0 and h_0 , $\left[x \left[\frac{l_0}{h_0} \right]'(x) h_0(x) \right]'(x) = 0$ and $\left[\frac{h_0}{x} \right]'(x) = 0$ so:

$$[x^2V''(x)]'_+(x^*)(G(x^*) - h_0(x^*)\bar{e}) = x^*\Sigma(x^*).$$

From expression of $\mathcal{E}(x^*)$:

$$h_0(x^*)[x^2V''(x)]'_-(x^*)\mathcal{E}(x^*) = x^*\Sigma(x^*),$$

$$h_0(x^*)[x^2V''(x)]'_+(x^*)(\mathcal{E}(x^*) - \bar{e}) = x^*\Sigma(x^*).$$

From concavity of G , $G(x) - xG'(x) \geq 0$ and $G''(x) < 0$,

hence $\Sigma(x) > 0$, so $[x^2V''(x)]'_-(x^*) > 0$ and $[x^2V''(x)]'_+(x^*) < 0$. \square

Proof of Proposition 2.5. In order to determine the behaviour of an optimal critical value x^* , we differentiate the Equation (6) with respect to growth rate r at $\lambda = 0$ to obtain:

$$\begin{aligned} \left[-G''_{x^2}(x^*, r) \frac{l_0}{h_0}(x^*) + (\delta - 2G'_x(x^*, r)) \left[\frac{l_0}{h_0} \right]'(x^*) - \left[\frac{l_0}{h_0} \right]''(x^*) G(x^*, r) \right] x^{*'}(r) &= \frac{\delta}{r} \frac{l_0}{h_0}(x), \\ \left[-G''_{x^2}(x^*, r) \frac{l_0}{h_0}(x^*) + (\delta + 2\left(\frac{G(x^*, r)}{x^*} - G'_x(x^*, r)\right)) \left[\frac{l_0}{h_0} \right]'(x^*) \right] x^{*'}(r) &= \frac{\delta}{r} \frac{l_0}{h_0}(x). \end{aligned}$$

From $G''_{x^2}(x, r) < 0$ and $G(0) = 0$, we deduce $G(x, r) - xG'_x(x, r) \geq 0$ and the left term in the brackets is positive, hence from positive marginal gain. \square

Proof of Proposition 2.6. For biomass x not close to K : $Q[V](x) = \int_X V(x(1 + z\epsilon)) dH(z)$,

$$\text{so: } [Q[V]]'(x) - V'(x) = \int_X [(1+z\epsilon)V'(x(1+z\epsilon)) - V'(x)]dH(z),$$

$$\begin{aligned} \text{where } V'(x(1+z\epsilon))(1+z\epsilon) - V'(x) &= z\epsilon(xV''(x) + V'(x)) \\ &+ \int_0^{z\epsilon} (z\epsilon - t)[V''((1+t)x)(1+t)x + V'((1+t)x)]'_t dt \\ &= z\epsilon(xV''(x) + V'(x)) \\ &+ \int_0^{z\epsilon} (z\epsilon - t)[V'''((1+t)x)(1+t)x^2 + 2V''((1+t)x)x]dt. \end{aligned}$$

For a sufficiently small ϵ , for all $x \neq x^*$ the integral can be approximated by $\frac{z^2\epsilon^2}{2}(x^2V'''(x) + 2xV''(x))$ (i.e. $\frac{z^2\epsilon^2}{2}[x^2V''(x)]'$) in the vicinity of critical biomass x^* . Hence at critical biomass x^* :

$$(i) \lim_{\epsilon \rightarrow 0} \frac{[Q[V]]'(x^*) - V'(x^*)}{\epsilon} = E[\mathcal{Z}](x^*V''(x^*) + V'(x^*)) = E[\mathcal{Z}] \left[x \frac{l_0}{h_0} \right]'(x^*),$$

$$(ii) \lim_{\epsilon \rightarrow 0} \frac{[Q[V]]'(x^*) - V'(x^*)}{\epsilon^2} = \frac{E[\mathcal{Z}^2]}{4}([x^2V''(x)]'_-(x^*) + [x^2V''(x)]'_+(x^*)).$$

From Proposition 2.4, $[x^2V''(x)]'_-(x^*) + [x^2V''(x)]'_+(x^*) = \frac{x^*\Sigma(x^*)}{h_0(x^*)} \frac{\bar{e} - 2\mathcal{E}(x^*)}{\bar{e} - \mathcal{E}(x^*)}$, hence using Proposition 2.3, $x'(0) > 0$ if $\mathcal{E}(x^*) < \frac{\bar{e}}{2}$ and $x'(0) < 0$ if $\mathcal{E}(x^*) > \frac{\bar{e}}{2}$. \square

Appendix C

Proof of Proposition 3.3. From total differentiation of Equation (12) with respect to growth rate r :

$$(l'_0(x^*(r)) - h'_0(x^*(r))V''_{xx}(x^*(r), r))x^{*'}(r) = h_0(x^*(r))V''_{xr}(x^*(r), r),$$

and using Equation (14) we deduce Equation (15). Let $A(x, r) = \frac{l_0}{h_0}(x) - V'_x(x, r)$.

In $\Omega - \{x|A(x, r) = 0\}$ the value function is smooth and is solution of:

$$\eta A(x, r)h_0(x)\bar{e} + V'_x(x, r)G(x, r) - (\delta + \lambda_x + \lambda_r)V(x, r) + \lambda.Q[V](x, r) = 0, \quad (19)$$

with $\eta = 0$ if $A(x, r) < 0$ (i.e. $x < x^*(r)$) and $\eta = 1$ if $A(x, r) > 0$ (i.e. $x > x^*(r)$).

Using Equation (14), from differentiation of Equation (19) with respect to biomass x and growth rate r at $\lambda_x = \lambda_r = 0$, at optimal critical value $x^*(r)$:

$$\begin{aligned} (G(x^*(r), r) - \eta h_0(x^*(r))\bar{e})V_{x^2r}''''(x^*(r), r) + V_{x^2}''(x^*(r), r)G_r'(x^*(r), r) + V_x'(x^*(r), r)G_{rx}''(x^*(r), r) &= 0, \\ (G(x^*(r), r) - \eta h_0(x^*(r))\bar{e})V_{x^2r}''''(x^*(r), r) + \frac{1}{r}\left(\frac{l_0}{h_0}\right)'(x^*(r))G(x^*(r), r) + \frac{l_0}{h_0}(x^*(r))G_x'(x^*(r), r) &= 0, \end{aligned}$$

then using Equation (13)

$$\begin{aligned} (G(x^*(r), r) - \eta h_0(x^*(r))\bar{e})V_{x^2r}''''(x^*(r), r) + \frac{\delta}{r}\frac{l_0}{h_0}(x^*(r)) &= 0, \\ h_0(x^*(r))(\mathcal{E}(x^*(r), r) - \eta\bar{e})V_{x^2r}''''(x^*(r), r) + \frac{\delta}{r}\frac{l_0}{h_0}(x^*(r)) &= 0. \end{aligned} \quad (20)$$

Hence, from positive marginal gain we deduce the expression of $V_{x^2r}''''^-(x^*(r))$ and $V_{x^2r}''''^+(x^*(r))$.

From total differentiation of Equation (15) with respect to growth rate r we deduce (17). \square

Proof of Proposition 3.4 (i) and (ii) Using the same reasoning than in the previous section and replacing λ by λ_x and $\lambda Q[V]$ by $\lambda_x Q_x[V]$ we obtain the same result than in the Proposition 2.6.

(iii) For growth rate r not close to \underline{r} and \bar{r} :

$$\begin{aligned} Q_r[V](x, r) - V(x, r) &= \int_{\underline{r}}^{\bar{r}} (V(x, r(1 + s\xi)) - V(x, r))dH_r(s), \\ \text{hence: } [Q_r[V]]'_x(x, r) - V'_x(x, r) &= \int_{\underline{r}}^{\bar{r}} [V'_x(x, r(1 + s\xi)) - V'_x(x, r)]dH_r(s) \end{aligned}$$

and:

$$V'_x(x, r(1 + s\xi)) - V'_x(x, r) = s\xi r V''_{xr}(x, r) + \int_0^{s\xi} (s\xi - t)r^2 V'''_{xr^2}(x, r(1 + t))dt.$$

For a sufficiently small ξ , the integral can be approximated by $\frac{s^2\xi^2}{2}r^2(V'''_{xr^2})_{\{-sign(s)\}}(x, r)$ in the vicinity of the critical biomass $x^*(r)$ for all growth rate r . Hence, due to Equation (15): $[Q_r[V]]'_x(x, r) - V'_x(x, r) = \frac{E[\mathcal{R}^2]\xi^2}{4}r^2(V'''_{xr^2}{}^-(x^*, r) + V'''_{xr^2}{}^+(x^*, r)) + O(\xi^3)$. From Equations (16) and (17), $V'''_{xr^2}{}^-(x^*, r) + V'''_{xr^2}{}^+(x^*, r)$ is proportional with the same sign to $\bar{e} - 2\mathcal{E}(x^*(r), r)$. \square

5 Figure Legends

Figure 1: Optimal evolution of biomass x with biomass updated

Figure 2: Optimal evolution of biomass x with biomass and growth rate updated